

REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

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1. AGENCY USE ONLY (Leave Blank)		2. REPORT DATE 2 June 2002		3. REPORT TYPE AND DATES COVERED Final -- 1 June 1999 -- 31 May 2002	
4. TITLE AND SUBTITLE Number Theoretic Methods in Harmonic Analysis: Theory and Application				5. FUNDING NUMBERS DAAD19-99-1-0240	
6. AUTHOR(S) Stephen D. Casey					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Department of Mathematics and Statistics, American University 4400 Massachusetts Ave., NW, Washington, DC 20016-8050				8. PERFORMING ORGANIZATION REPORT NUMBER OSP Control Number LRW019901.1	
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) U. S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211				10. SPONSORING / MONITORING AGENCY REPORT NUMBER DAAD19-99-1-0240 40076.8-MA	
11. SUPPLEMENTARY NOTES The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.					
12 a. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; distribution unlimited.				12 b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) We have used tools from theory of harmonic analysis and number theory to extend existing theories and develop new approaches to problems. This work has focused on two areas. We have developed algorithms on extensions of Euclidean domains which have led to new computationally straightforward algorithms for parameter estimation for periodic point processes, and in particular, for sparse, noisy data. We have shown why Fourier analytic methods, e.g., Wiener's periodogram, do not produce maximum likelihood estimates for the sparse data sets on which our methods work. We are also working on extending our work to multiply periodic processes. We have also used the tools from multichannel deconvolution to develop a new procedure for multi-rate sampling. We have investigated applying these techniques to develop a new procedure for A--D conversion. We have also developed interlinked wavelet bases, interlinked via number-theoretic conditions that proved useful for both multi-channel deconvolution and multi-rate sampling. We have also created new procedures for sampling in radial domains which have application to radar and sonar. We have also extended these ideas to operator theory, creating sets of strongly coprime chirp and chirplet operators.					
14. SUBJECT TERMS Deconvolution, Sampling, Wavelets, Parameter Estimation, Periodic Process				15. NUMBER OF PAGES 17	
				16. PRICE CODE	
17. SECURITY CLASSIFICATION OR REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION ON THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL		

NSN 7540-01-280-5500

Standard Form 298 (Rev.2-89)
Prescribed by ANSI Std. Z39-18
298-102

20030515 199

Report Document Page (SF 298)
(Continuation Sheets)

Final Progress Report for
NUMBER THEORETIC METHODS IN HARMONIC ANALYSIS:
THEORY AND APPLICATION
ARO Grant Number DAAD19-99-1-0240
Stephen D. Casey
Department of Mathematics and Statistics
American University

June 1st, 1999 – May 31th, 2002

1.) FORWARD

We have used tools from theory of harmonic analysis and number theory to extend existing theories and develop new approaches to problems. This work has focused on several areas. We have developed algorithms on extensions of Euclidean domains which have led to new computationally straightforward algorithms for parameter estimation for periodic point processes, and in particular, for sparse, noisy data. We have shown why Fourier analytic methods, e.g., Wiener's periodogram, do not produce maximum likelihood estimates for the sparse data sets on which our methods work. We are also working on extending our work to multiply periodic processes. We have also extended our work on *multichannel deconvolution*. We have used the tools from multichannel deconvolution to develop a new procedure for multi-rate sampling. We have investigated applying these techniques to develop a new procedure for A-D conversion. We have also developed interlinked wavelet bases, interlinked via number-theoretic conditions that proved useful for both multi-channel deconvolution and multi-rate sampling. We have also created new procedures for sampling in radial domains which have application to radar and sonar. We have also extended these ideas to operator theory, creating sets of strongly coprime chirp and chirplet operators.

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3.) STATEMENTS OF PROBLEMS STUDIED

Our work on multichannel deconvolution included the following.

- i.) Fit our general models of system kernels with models of currently deployed active and passive remote sensors as convolution equations. Then apply our theory to develop systems in which complete signal information can be recovered. In particular, we are looking to apply the theory in general imaging systems (e.g., for sub-pixel resolution), radar (e.g., strongly coprime chirp pulses), etc. (see [2], [8], [38]).
- ii.) Develop a basis in which to develop the multichannel theory which is more amenable to discretization. This would then be directly applicable to (i.) (see [2], [8]).
- iii.) Coordinate the theory with filtering systems for the removal of noise and/or other unwanted information (see [15]).
- iv.) Coordinate the theory with irregular sampling theory, and wavelet and Gabor analysis (See [11], [12], [10], [2]).

We also continued our work on parameter estimation.

- v.) Develop computationally straightforward techniques for spectral analysis of a very broad class of periodic processes, including procedures so that estimates achieve the Cramer-Rao bound (see [9]).
- vi.) Extend these techniques to the complete analysis of multiply periodic point processes, including the recovery of the fundamental period(s), phase information, the multiples of the periods, and the deinterleaving of the data (see [7]).

Additional items include new results on derivatives and monotonicity (see [5], [6]) and new procedures for visualizing complex mappings (see [3]).

4.) SCIENTIFIC PROGRESS AND ACCOMPLISHMENTS

Sampling on Unions of Non-Commensurate Lattices via Complex Interpolation Theory [11], [12]

MAIN RESULTS

Solutions to the analytic Bezout equation associated with certain multichannel deconvolution problems are interpolation problems on unions of non-commensurate lattices. These solutions provide insight into how one can develop general sampling schemes on such sets. We give specific examples of non-commensurate lattices, and use a generalization of B. Ya. Levin's sine-type functions to develop interpolating formulae on these sets. Let $t \in \mathbb{R}$, and let α be an irrational that is poorly approximated by rationals, e.g., $\sqrt{2}$, or $(1 + \sqrt{5})/2$. Let $\mu_1(t) = \chi_{[-1,1]}(t)$, $\mu_2(t) = \chi_{[-\alpha,\alpha]}(t)$ model the impulse response of the channels of a two-channel system. Then $\widehat{\mu_1}(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta}$, $\widehat{\mu_2}(\zeta) = \frac{\sin(2\pi\alpha\zeta)}{\pi\zeta}$. Since α is poorly approximated by rationals, $\{\mu_i\}$ is strongly coprime. Now let

$$\Gamma_1 = \left\{ \frac{\pm k}{2} \right\}, \Gamma_2 = \left\{ \frac{\pm k}{2\alpha} \right\} \text{ for } k \in \mathbb{N}, \text{ and } \Gamma = \Gamma_1 \cup \Gamma_2.$$

Note that the information contained in the original signal is reconstructed by creating deconvolvers defined initially on $\Gamma \cup \{0\}$. Order the elements of Γ , denoting this as $\Gamma = \{\lambda_k\}$. We have the following.

Theorem 1 *Let α be an irrational that is poorly approximated by rationals, and let f be a $(1+\alpha)$ -band-limited function. Then f is uniquely determined by $\{f(\lambda_k)\} \cup \{f(0), f'(0)\}$.*

Note, for a $(1+\alpha)$ -band-limited function, the Nyquist rate is $1/(2(1+\alpha))$. However, our individual sampling rates are $1/2$ and $1/(2\alpha)$. Both these rates are below Nyquist. The reconstruction of f from this lattice is achieved by using complex interpolation theory. These techniques go back to basic Lagrange interpolation, and were developed for entire functions by various mathematicians. We first solve the problem on $\Gamma \cup \{0\}$. Let $G(z) = \sin(2\pi z) \sin(2\pi\alpha z)$. Then, $G(z)$ is an entire function, which is almost periodic on \mathbb{R} , and has simple zeros on Γ , and a double zero at $\{0\}$.

Proposition 1 *Let $\lambda_k \in \Gamma$ and let*

$$H_k(z) = \frac{G(z)}{G'(z)(z - \lambda_k)}, \text{ and so } H_j(\lambda_k) = \delta_{jk}.$$

At $z = 0$, we have to construct interpolating functions K_1, K_2 so that

$$K_1(0) = 1, K_1'(0) = 0, K_2(0) = 0, K_2'(0) = 1.$$

Using the Taylor expansion of G , we derive the following.

Proposition 2

$$K_1(z) = \frac{G(z)}{(G''(0)/2!)(z^2)} = \frac{G(z)}{(4\pi^2\alpha)(z^2)}, K_2(z) = \frac{G(z)}{(G''(0)/2!)(z)} = \frac{G(z)}{(4\pi^2\alpha)(z)}.$$

Combining these two propositions gives us the reconstruction formula.

Theorem 2 *Let α be an irrational that is poorly approximated by rationals, and let f be a $(1+\alpha)$ -band-limited function. Let $\Gamma_1 = \{\frac{\pm k}{2}\}$, $\Gamma_2 = \{\frac{\pm k}{2\alpha}\}$, for $k \in \mathbb{N}$, and let $\Gamma_1 \cup \Gamma_2 = \Gamma = \{\lambda_k\}$. Then f is uniquely determined by $\{f(\lambda_k)\} \cup \{f(0), f'(0)\}$. Moreover, f can be approximated from its values on $\Gamma \cup \{0\}$ by the formula*

$$f(t) \approx \sum_{\lambda_k \in \Gamma} f(\lambda_k) \frac{G(t)}{G'(\lambda_k)(t - \lambda_k)} + f(0) \frac{G(t)}{(4\pi^2\alpha)(t^2)} + f'(0) \frac{G(t)}{(4\pi^2\alpha)(t)},$$

where

$$G(t) = \sin(2\pi t) \cdot \sin(2\pi\alpha t).$$

We need to point out two items. Let $r_1 = 1, r_2 = \alpha$. First, the sampling grid is rigid. Perturbation of the grid results in a loss of information. Second, that because sampling points in Γ can get arbitrarily close together ($\inf\{|\lambda_m - \lambda_n|\} = 0$, and so $\inf\{\prod_{k \neq j} |\sin(2\pi r_k \lambda)| : \lambda \in \Gamma_j\} = 0$), the dual elements of the interpolating functions can not form a Riesz basis and the interpolating formula can not converge in norm. In fact, the dual elements of the interpolating functions do form a Bessel sequence, but do not form a frame and therefore do not form a Riesz basis. The problems occur at points where the sampling points get close together. The interpolating function follows the original function along exactly, except for a very subtle "ripple" at those points where the sampling points get close together. We are currently exploring ways to stabilize the construction. The first involves continuously backing off the bandwidth. This works, but we need to set up exact bounds.

The second approach involves rewriting Γ in terms of the “separated” and “close” points. At the close points, we may be able to rewrite the formulae in terms of f' .

The result generalizes. We can create sampling sets on ℓ lattices using a set of numbers $\{r_i\}_{i=1}^{\ell}$ such that (r_i/r_j) is poorly approximated by rationals for $i \neq j$. Again, if $\{p_i\}_{i=1}^{\ell-1}$ is a set of primes,

$$\{1, \sqrt{p_1}, \sqrt{p_1 p_2}, \dots, \sqrt{p_1 p_2 \cdots p_{\ell-1}}\}$$

is a set of numbers whose ratios are poorly approximated by rationals. Let $\Gamma_k = \left\{ \frac{\pm n}{2r_k} \right\}$ for $n \in \mathbb{N}$, and let

$$\bigcup_{k=1}^{\ell} \Gamma_k = \Gamma = \{\lambda_k\}.$$

We reconstruct on $\Gamma \cup \{0\}$, letting

$$G(z) = \prod_{k=1}^{\ell} \sin(2\pi r_k z) \quad \text{and letting} \quad H_m(z) = \frac{G(z)}{G'(z)(z - \lambda_m)}.$$

Then $H_m(\lambda_n) = \delta_{mn}$.

The interpolating functions at the origin are a linear combination of $g_j = \frac{G(z)}{(z^j)}$, $j = 1, \dots, \ell$, chosen so that $K_k^{(j-1)}(0) = \delta_{kj}$, $k, j = 1, \dots, \ell$. We can also create these as unions of regular lattices in higher dimensions.

We close the paper by proving that, from a measure-theoretic and thus probabilistic point of view, the strongly coprime condition on which our constructions lie is quite natural. The numbers in the complement of our condition are called the Liouville numbers \mathbb{L} . We show the following.

Theorem 3 *The class of Liouville numbers \mathbb{L} has Lebesgue measure zero.*

Thus if one chooses an irrational number at random (say, using the uniform distribution on any finite interval), one would almost surely choose an irrational which is poorly approximated by rationals. However, the situation is even better than this.

Theorem 4 *Let \mathbb{L} denote the class of Liouville numbers. Then, for every $\sigma > 0$, \mathbb{L} has σ -dimensional Hausdorff measure zero.*

New Directions in Sampling and Multi-Rate A-D Conversion Via Number Theoretic Methods [10]

MAIN RESULTS

This paper uses the new sampling theory developed in the paper above to provide a new approach to multi-rate A-D conversion. This uses multiple converters with different rates. Consider a $1 + \alpha$ bandlimited signal, with Nyquist sampling rate $f_N = 2(1 + \alpha)$ (samples / sec). Let the sampling rates be denoted f_i , $i = 1, \dots, K$. These rates are such that (i) their ratios are strongly coprime, and (ii) $\frac{1}{K} \sum_{i=1}^K f_i = f_N$, (the average of the rates is equal to the Nyquist rate).

As above, the signal may be reconstructed from two samplers running at rates $f_1 = 2$, and $f_2 = 2\alpha$. We emphasize that this choice of f_1 is arbitrary and may take any value so long as the

above conditions are satisfied. While f_1 remains fixed at a predetermined rate, $f_2 \rightarrow f_N$ from below as α becomes large. Thus, the introduction of the *fixed* low rate sampler at f_1 has relaxed the faster sampling rate below the Nyquist rate. Because the multiple rates are non-commensurate, they will eventually yield sampling points that are arbitrarily close to one another, within $\delta > 0$ say. In this case, there exists an $\epsilon > 0$ such that one of the samples occurring within the δ neighborhood may be omitted, while guaranteeing unique reconstruction for signals bandlimited to $1 + \alpha - \epsilon$. It is also possible to look at reconstruction f' locally.

The multi-rate sampling approach can be extended to more than two rates. Consider a $1 + \alpha + \beta$ bandlimited signal, with α and β selected to satisfy the coprime condition (for example, choose α and β to be square roots of products of primes). Now, three parallel samplers may be used with rates $f_1 = 1/2$, $f_2 = 1/(2\alpha)$, and $f_3 = 1/(2\beta)$. (Again we emphasize that these choices are not unique.) We consider two cases. First, if $\alpha \approx \beta$, then f_2 and f_3 approach $f_N/2$ from below as α increases. Second, if $\beta \approx k\alpha$, for $k > 0$ integer, then $f_2 \rightarrow f_N/(1+k)$, while $f_3 \rightarrow f_N k/(1+k)$ as α grows. Thus, many combinations of slow and fast rates are possible.

It is interesting to contrast this scheme with the more conventional case of K interleaved identical parallel A-Ds, sampling at the same rate, $f_i = f_N/K$. The coprime multi-rate approach allows the A-Ds to run at different rates, without the need for signal preconditioning, e.g., bandpass filtering. Thus, the proposed multi-rate scheme generalizes the conventional interleaved case. Also, the slower A-Ds could utilize more quantization levels. The multi-rate approach requires non-commensurate rates, so a scheme must be developed for generating these rates while maintaining the underlying synchronism. This requires close approximation to irrationals from rational numbers. In addition, the multiple rates will cause sampling times to become arbitrarily close. However, as noted above, when neighboring samples fall into a δ neighborhood, one sample may be omitted (with a corresponding ϵ small reduction in the achievable signal bandwidth).

Sinusoidal Frequency Estimation via Sparse Zero Crossings [9]

MAIN RESULTS

We consider estimation of the period of a sinusoid in additive Gaussian noise, based on observations of the zero-crossing (ZC) times. The problem is treated in a continuous-time framework. It is assumed that the signal-to-noise ratio is sufficient (roughly ≥ 8 dB) such that the noise may be approximated as additive in the phase. An exact mean-square error analysis is provided for this approximation. We apply modified Euclidean algorithms (MEA's) and their least-squares refinements in this framework, to estimate the period of the sinusoid, with low complexity. Unlike linear regression methods based on phase samples, the proposed approach works with very sparse ZC measurements, and is resistant to outliers. The MEA-based approach is motivated by the fact that, in the noise-free case, the greatest common divisor (gcd) of a sparse set of the first differences of the zero crossing times is very highly likely to be the half-period of the sinusoid. The MEA acts to robustly estimate the gcd of the observed noisy data. The MEA period estimate may be refined via a least-squares approach, that asymptotically achieves the appropriate Cramer-Rao bound. Simulation results illustrate the algorithms with as few as 10 zero-crossing times. The algorithm behavior is also studied using Bernoulli and random burst models for the missing ZC times, and good performance is demonstrated with very sparse observations.

The observed data is modeled as a time series – a sampled sinusoid in Gaussian noise, expressed as $x(n) = A \exp i(\omega_0 n \rho + \phi) + z(n)$, for $n = n_0, n_0 + 1, \dots, n_0 + N - 1$. The noise $z(n)$ is assumed

to be a white Gaussian zero-mean complex noise sequence with variance σ_z^2 . The amplitude A , frequency ω_0 , and phase ϕ are regarded as unknown constants for a particular realization. The value ρ is the sampling period. We can write

$$\begin{aligned} x(n) &= A \exp i(\omega_0 n \rho + \phi) + z(n) = \left[1 + \frac{z(n)}{A} \exp(-i(\omega_0 n \rho + \phi)) \right] A \exp(i(\omega_0 n \rho + \phi)) \\ &= [1 + \nu(n)] A \exp(i(\omega_0 n \rho + \phi)), \end{aligned}$$

where $\nu(n) = \frac{z(n)}{A} \exp[-i(\omega_0 n \rho + \phi)]$. Note that $E[\nu(n)] = 0$ and $\text{var}\{\nu(n)\} = \sigma_z^2/A^2 = \text{SNR}^{-1}$, so $\nu(n) \rightarrow 0$ as $\text{SNR} \rightarrow \infty$. If we split ν into real and imaginary parts, writing $\nu = w + iu$, we can express $1 + \nu(n)$ in terms of modulus and phase as

$$1 + \nu(n) = \left[[1 + w(n)]^2 + [u(n)]^2 \right]^{\frac{1}{2}} \exp \left(i \arctan \left[\frac{u(n)}{1 + w(n)} \right] \right).$$

We give two proofs of the following result in the paper.

Theorem 5 *If $A^2/\sigma_z^2 \gg 1$, (i.e., high SNR), then $[1 + \nu(n)] \approx \exp[iu(n)]$. This in turn gives*

$$x(n) = A \exp i(\omega_0 n \rho + \phi) + z(n) \approx A \exp(i(\omega_0 n \rho + \phi + u(n))).$$

Residue and Sampling Techniques in Deconvolution [8]

MAIN RESULTS

In this paper we will study techniques for solving the following general problem, which we will refer to as the *Multisensor Deconvolution Problem* or MDP: *Given a collection of compactly supported distributions, $\{\mu_i\}_{i=1}^m \subseteq \mathcal{E}'(\mathbb{R}^d)$, how can we recover an arbitrary function $f \in C^\infty(\mathbb{R}^d)$ from the data $\{s_i\}_{i=1}^m = \{f * \mu_i\}_{i=1}^m$?* The solution technique that we consider involves the construction of *deconvolvers*, which come in a variety of types but which are essentially a collection of distributions which (1) depend only on the convolvers $\{\mu_i\}_{i=1}^m$ and (2) allow for the solution of the MDP with only simple linear operations on the data $\{s_i\}_{i=1}^m$.

We describe a solution to a modified analytic Bezout equation. The technique of solution involves the Jacobi interpolation formula and the Cauchy residue calculus. We create an approximate deconvolution system by solving the Bezout equation $\sum_{i=1}^m \hat{\mu}_i \cdot \hat{\nu}_{i,\varphi} = \hat{\varphi}$, where, for $0 < \epsilon < 1$, $\varphi(x) = \psi_\epsilon(x) = \epsilon^{-1} \psi(\epsilon^{-1}x)$. Thus, $\hat{\psi}_\epsilon \rightarrow 1$ as $\epsilon \rightarrow 0^+$. This gives us our deconvolving functions. Then, as $\varphi \rightarrow \delta$, $f * \varphi \rightarrow f$ in the sense of distributions. That is, the signals s_i are filtered by the $\nu_{i,\varphi}$ (which have been created digitally, optically, etc., in coordination with the creation of the system and possibly tailored to be optimized under some constraint) and added, resulting in the reconstruction of $f * \varphi$. By controlling ϵ , f can be reconstructed to any predetermined accuracy. We also create these solutions by using techniques from real-variable sampling theory.

On Positive Derivatives and Monotonicity [5], [6]

MAIN RESULTS

A fundamental result of calculus states that a function f with a positive derivative Df on an interval is increasing on that interval. The result follows directly from the Mean Value Theorem. We explore the extent to which the hypotheses of the Mean Value Theorem can be weakened and f still shown to be increasing. Let f be a continuous function on an interval. By constructing counterexamples using Cantor sets, we show that the assumption $Df > 0$ a.e. does not imply that f is increasing. We can, however, show that if $Df > 0$ except on a countable subset of an interval, then f is increasing. We call this the Countable Exceptional Set Theorem. This theorem is generalized by the Goldowsky-Tonelli Theorem, which tells us that if Df exists except on a countable subset of an interval and $Df > 0$ a.e., then f is increasing. However, we then go on to show, in a very natural sense, that Goldowsky-Tonelli is a vacuous extension of the Countable Exceptional Set Theorem. We close by exploring a few related questions, asking to what extent monotonicity implies positive derivative.

Complex Analysis *An Animated Approach* [3]

SAMPLE LESSONS

Sample Lesson 1: An Introduction to Animated Graphs

One of the most fundamental skills needed in understanding complex analysis is visualizing complex functions. This is, however, a challenging task. The graph of a complex function $f : \mathbb{C} \rightarrow \mathbb{C}$ lives in $\mathbb{C} \times \mathbb{C}$ (two complex dimensions), which is four real dimensions. Given that we only live in three spatial dimensions, visualization is difficult. We can, however, get around this by using our fourth dimension – time – as a tool to help us visualize complex functions. We will watch a given function evolve in time, starting from the unaltered complex plane and ending with the range of the function. The computer provides us with an ideal tool to see this evolution. We compute a frame-by-frame movie of the evolution “off-line,” and show it at relatively high speeds, giving the appearance of a continuous deformation. These “movies” of complex functions will be called *animated graphs*.

A mathematical framework for animated graphs already exists. This deformation of the function is a *homotopy*, from the identity to the target function. We will give a formal definition of homotopy elsewhere. For now, we can think of these graphs as movies in time t , $0 \leq t \leq T$, where we are looking at $F(z, t) = (1 - t)z + tf(z)$. Since a complex function could have a domain as large as the entire complex plane, we will highlight some of the interesting curves and arcs in the domain.

The animated graphs were developed using $f(z)$, a software package for complex variables written by Martin Lapidus of *Lascaux Graphics*. The animated graphs are programs developed by the author within $f(z)$, just as one would write Notebooks inside of *Mathematica*. The animated graphs consist of curves and arcs in the domain on the left and their images under mapping on the right. However, it is very important to note that you will not just see the static images, but rather a “continuous” (actually, frame-by-frame) deformation of the curves and arcs in the domain to their corresponding images in the range. It is also important to note that this is an extension of the traditional representation of complex mappings, namely by drawing curves, arcs, and or regions in the domain, and then showing their images in the range.

We also have a set of animated graphs for linear mappings, and will expand upon the matrix model of \mathbb{C} . The animated graphs will include linear mappings, and will demonstrate that the

action of multiplying by a complex number is the same as the action of a 2×2 matrix on the vector z . The animations of the linear maps show the actions of the maps on rectangular grids. For example, the mapping $iz = e^{i\pi/2}re^{i\theta} = re^{i\theta+\pi/2}$ rotates a complex number counterclockwise by $\pi/2$. Note that this is the same result as multiplying the complex vector $z = (x, y)$ by the matrix

$$M_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Sample Lesson 2: Local Behavior of Polynomial and Rational Functions

This second group of sample lessons is included to show that the animated graphs can be used to effectively demonstrate certain principles. An important concept in complex variables is that, thanks to the power of Taylor and Laurent expansions, the behavior of an analytic or meromorphic function in a neighborhood of a zero or a pole can be understood by factoring the function. For example, let f be an analytic function which has a zero of order k at z_0 . In a sufficiently small neighborhood \mathcal{N} of z_0 , f can be written as $f(z) = (z - z_0)^k F(z)$, where F is analytic and non-zero in \mathcal{N} . Moreover, the behavior of the mapping f in \mathcal{N} is essentially $f(z) \approx (z - z_0)^k F(z_0)$, that is, the mapping looks like the polynomial $(z - z_0)^k$ multiplied by the complex number $F(z_0)$. This important principle is demonstrated for the functions $f(z) = (z - (1+i))(z - (\frac{1}{2} - i\frac{1}{2}))$ and $f(z) = z^4 - 1$. The animations show how the function maps neighborhoods of the zeros onto a region in a neighborhood of the origin in the range. Even though the circles and line segments in the neighborhoods of zeros end up on the same curves in the image, the paths they take to get there are quite different. The different magnifications and rotations that the circles and line segments undergo are the result of multiplying the map $(z - z_0)$ by the appropriate $F(z_0)$.

Similarly, let g be a meromorphic function which has a pole of order l at w_0 . In a sufficiently small neighborhood \mathcal{M} of w_0 , g can be written as $g(z) = (z - w_0)^{-l} G(z)$, where G is analytic and non-zero in \mathcal{M} . Moreover, the behavior of the mapping g in \mathcal{M} is essentially $g(z) \approx (z - w_0)^{-l} G(w_0)$, that is, the mapping looks like the function $(z - w_0)^{-l}$ multiplied by the complex number $G(w_0)$. This important principle is demonstrated for the functions $g(z) = \frac{1}{z - (1+i)}$ and $g(z) = \frac{1}{z^4 - 1}$.

Let f be a non-trivial analytic function in a region Ω with a zero of order k at $z_0 \in \Omega$. Thus by the Taylor expansion of f , $f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$, because $f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{(k-1)}(z_0) = 0$. Also, since zeros of a non-trivial analytic function may not cluster, there exists a neighborhood \mathcal{N} in which we may write $f(z) = ((z - z_0)^k) F(z)$, where F is a non-zero analytic function in \mathcal{N} such that $\lim_{z \rightarrow z_0} F(z) = \frac{f^{(k)}(z_0)}{k!} \neq 0$. We get the Taylor series for F by factoring $F(z) = \sum_{n=0}^{\infty} \frac{f^{(n+k)}(z_0)}{(n+k)!} (z - z_0)^n$. Therefore, in \mathcal{N} , the behavior of the mapping f is essentially $f(z) \approx (z - z_0)^k F(z_0)$, that is, the mapping looks like the polynomial $(z - z_0)^k$ multiplied by the complex number $F(z_0)$.

We will first demonstrate this important principle for the function $f(z) = (z - (1+i))(z - (\frac{1}{2} - i\frac{1}{2}))$ has two simple zeros. In a neighborhood of $1+i$, $f(z) \approx (\frac{1}{2} + i\frac{3}{2})(z - (1+i)) = f'(1+i)(z - (1+i))$. In a neighborhood of $(\frac{1}{2} - i\frac{1}{2})$, $f(z) \approx (-\frac{1}{2} + i\frac{3}{2})(z - (\frac{1}{2} - i\frac{1}{2})) = f'(\frac{1}{2} - i\frac{1}{2})(z - (\frac{1}{2} - i\frac{1}{2}))$.

The function $f(z) = z^4 - 1$ has a little more interesting behavior. Neighborhoods of each of the four simple zeros $1, i, -1, -i$ get mapped to neighborhoods of the origin. However, each of these neighborhoods takes a different path to get there. We can compute the different rotations and magnifications that each neighborhood undergoes by factoring. If we index the zeros as $z_1 = 1$, $z_2 = i$, $z_3 = -1$, $z_4 = -i$, then $f(z) = (z - z_i)F(z_i) = (z - z_i)f'(z_i)$, where $F(z_1) = (z_1)^3 + (z_1)^2 + (z_1) + 1 = 4 = f'(z_1)$, $F(z_2) = (z_2)^3 + i(z_2)^2 - (z_2) - i = -4i = f'(z_2)$, $F(z_3) = (z_3)^3 - (z_3)^2 +$

$(z_3) - 1 = -4 = f'(z_3)$, $F(z_4) = (z_4)^3 - i(z_4)^2 - (z_4) + i = 4i = f'(z_4)$. This same behavior extends to general analytic functions.

Similarly, let g be a meromorphic function which has a pole of order l at w_0 . Using the Laurent expansion in a deleted neighborhood \mathcal{M} of w_0 , we have $g(z) = \sum_{n=0}^{\infty} a_n(z-w_0)^n + \sum_{m=1}^l b_m(z-w_0)^{(-m)}$, where the coefficients are given by the integrals $a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{(z-w_0)^{n+1}} dz$, $b_m = \frac{1}{2\pi i} \int_{\Gamma} g(z)(z-w_0)^{m-1} dz$, for $\Gamma = \{z : |z-w_0| = r\}$ contained in \mathcal{M} . Of course, we can use the techniques given in section 6.4 to compute the coefficients. We can once again factor this expansion, writing g as $g(z) = ((z-w_0)^{(-l)})G(z)$, where G is a non-zero analytic function in \mathcal{M} such that $\lim_{z \rightarrow w_0} G(z) = b_l \neq 0$. The Taylor series for G is $G(z) = \sum_{m=1}^l b_m(z-w_0)^{(l-m)} + \sum_{n=0}^{\infty} a_n(z-w_0)^{(n+l)}$. Therefore, in \mathcal{M} , the behavior of the mapping g is essentially $g(z) \approx (z-w_0)^{-l}G(w_0)$, that is, the mapping looks like the rational function $(z-w_0)^{(-l)}$ multiplied by the complex number $G(w_0)$.

This important principle is demonstrated for the function $g(z) = \frac{1}{z^4-1}$, which has four simple poles at $i, i, -1, -i$. We can compute the different rotations and magnifications that each neighborhood undergoes by factoring. If we index the poles as $w_1 = 1, w_2 = i, w_3 = -1, w_4 = -i$, then $g(z) = \frac{1}{(z-w_i)}G(w_i)$, where $G(w_1) = \frac{1}{(w_1)^3+(w_1)^2+(w_1)+1} = \frac{1}{4}$, $G(w_2) = \frac{1}{(w_2)^3+i(w_2)^2-(w_2)-i} = \frac{1}{-4i} = \frac{i}{4}$, $G(w_3) = \frac{1}{(w_3)^3-(w_3)^2+(w_3)-1} = \frac{-1}{4}$, $G(w_4) = \frac{1}{(w_4)^3-i(w_4)^2-(w_4)+i} = \frac{1}{4i} = \frac{-i}{4}$.

This same behavior extends to general meromorphic functions. However, the situation is quite different in a neighborhood of an essential singularity. Recall that the Casorati-Weierstrass Theorem told us that a deleted neighborhood of an essential singularity gets mapped densely onto the entire complex plane. The behavior is actually more amazing. One of the theorems we will prove is the *Great Picard Theorem*, which tells us that if a function g is analytic in a deleted neighborhood of a point w_0 and has an essential singularity at w_0 , then in every deleted neighborhood of w_0 , g assumes every complex number, with one possible exception, an infinite number of times.

Systems of Convolution Equations, Deconvolution, Shannon Sampling, and the Wavelet and Gabor Transforms [2]

See enclosed table of contents.

5.) LIST OF MANUSCRIPTS

Published

Refereed Articles

(With B. Sadler) "Sinusoidal frequency estimation via sparse zero crossings," *Jour. Franklin Inst.*, No. 337, 131-145 (2000).

(With B. Sadler) "New directions in sampling and multi-rate A-D conversion via number theoretic methods," *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2000)*, Vol. 3, 1417-1420 (2000) (refereed).

"Sampling and reconstruction on unions of non-commensurate sampling lattices via complex interpolation theory," *1999 International Workshop on Sampling Theory and Applications*, 48-51 (1999) (refereed).

(With B. Sadler) "Sampling on unions of non-commensurate lattices via complex interpolation theory," *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP '99)*, Vol. 3, 1641–1644 (1999) (refereed).

Book Chapters

(With D. Walnut) "Residue and sampling techniques in deconvolution," Chapter 9 in *Modern Sampling Theory: Mathematics and Applications*, Birkhauser Research Monographs, ed. by P. Ferreira and J. Benedetto, 193–217 (2001).

Accepted for Publication

(With R. Holzsager) "On positive derivatives and monotonicity," accepted for *Missouri Journal of Mathematical Sciences*, 13 pp. (2002).

Submitted for Publication

(With R. Holzsager) "Monotonicity with/without mean values" submitted to *Mathematics Magazine*, 17 pp.

Books in Progress

(With C. A. Berenstein and D. F. Walnut) *Systems of convolution equations, deconvolution, Shannon sampling, and the wavelet and Gabor transforms*, research monograph for *SIAM Monographs on Mathematical Modeling and Computation* (invited).

Complex Analysis (An Animated Approach), proposed text for Addison–Wesley Advanced Text Series.

Technical Reports

ARO Interim Progress Report for *Number Theoretic Methods in Harmonic Analysis: Theory and Application* – January 2002.

ARO Interim Progress Report for *Number Theoretic Methods in Harmonic Analysis: Theory and Application* – January 2001.

ARO Interim Progress Report for *Number Theoretic Methods in Harmonic Analysis: Theory and Application* – January 2000.

Presentations

National Institute of Science and Technology – "Complex mappings from an evolutionary viewpoint (A new way to see some classical mathematics)," – April 2002 (1 hour presentation).

James Madison University Mathematics Department Colloquium – "Pi, the Primes, Periodicities, and Probability," – March 2002 (1 hour presentation).

New Mexico Analysis Seminar – "Sampling on non-commensurate lattices via complex interpolation theory," – February 2002 (45 minute presentation).

Morgan State University Mathematics Department Colloquium – “Pi, the Primes, Periodicities, and Probability,” – September 2001 (1 hour presentation).

American University Math/Stat Department Colloquium – “Complex mappings from an evolutionary viewpoint (A new way to see some classical mathematics),” – April 2001 (1 hour presentation).

University of Maryland, Baltimore County Mathematics Department Colloquium – “Pi, the Primes, Periodicities, and Probability,” – March 2001 (1 hour presentation).

University of Maryland Statistics Seminar – “Pi, the Primes, Periodicities, and Probability,” – October 2000 (1 hour presentation).

U.S. Naval Academy Mathematics Department Colloquium – “Complex mappings from an evolutionary viewpoint (A new way to see some classical mathematics),” – September 2000 (1 hour presentation).

Joint Meeting of the American Mathematical Society and the Mathematical Association of America, Washington, DC – special session on “Recent Advances in Harmonic and Complex Analysis” (co-chair with C.A. Berenstein, B.Q. Li, D.F. Walnut, C.C. Yang) – “Analytic Bezout equations, sampling, and the strongly coprime condition” – January 2000 (20 minute presentation).

(with D. Bell) – Meeting of the Mathematical Association of America, Loyola University “Visualizing complex mappings from an evolutionary point of view” – November 1999 (2 hour workshop).

American University Student-Faculty Colloquium – “Deconvolution and sampling via number theoretic chicanery,” – October 1999 (1 hour presentation).

1999 Workshop on Sampling Theory and Applications, Loen, Norway – “Sampling and reconstruction on non-commensurate lattices via complex interpolation theory,” Session on “Direct Sampling Methods and Image Reconstruction” – August 1999 (30 minute presentation).

1999 Workshop on Sampling Theory and Applications, Loen, Norway – “How natural is the strongly coprime condition?” Session on “Direct Sampling Methods and Image Reconstruction” – August 1999 (30 minute presentation).

6.) SCIENTIFIC PERSONNEL – PROMOTIONS, HONORS, AND AWARDS

Promoted to rank of Full Professor – May 2001.

Finalist for Teacher of the Year at American University (Student Award) – (For Academic Year 2000-2001).

Selected as Program Chair Elect of the Mathematical Association of America MD-VA-DC Section.

American University Faculty Award for Outstanding Teaching – For Academic Year 1998-1999).

Editorial Boards

Associate Editor, *Sampling Theory in Signal and Image Processing*.

7.) REPORT OF INVENTIONS

N/A

8.) BIBLIOGRAPHY

References

- [1] **Proposer's Papers Related to Grant**
- [2] Berenstein C. A., Casey, S. D., and Walnut, D. F., "Systems of convolution equations, deconvolution, Shannon sampling, and the wavelet and Gabor transforms," research monograph for *SIAM Monographs on Mathematical Modeling and Computation* (invited), in progress.
- [3] Casey, S. D., "Complex Analysis (An Animated Approach)," proposed text for Addison-Wesley Advanced Text Series, in progress.
- [4] Berenstein C. A., Casey, S. D., and Walnut, D. F., "Exact multichannel deconvolution," survey article for *Advances in Imaging and Electron Physics* (invited), in progress.
- [5] Casey, S. D., and Holzsager, R. A., "When does a positive derivative guarantee monotonicity?" submitted to *Mathematics Magazine*, 17 pp. (2002).
- [6] Casey, S. D., and Holzsager, R. A., "On positive derivatives and monotonicity," to appear in *Missouri Journal of Mathematical Sciences*, 13 pp. (2002).
- [7] Casey, S. D., "Variations on Weyl's equidistribution theorem with applications to deinterleaving," 28 pp., preprint (2002).
- [8] Casey, S. D., and Walnut, D. F., "Residue and sampling techniques in deconvolution," Chapter 9 in *Modern Sampling Theory: Mathematics and Applications*, Birkhauser Research Monographs, ed. by P. Ferreira and J. Benedetto, 193-217 (2001).
- [9] Sadler, B. M., and Casey, S. D., "Sinusoidal frequency estimation via sparse zero crossings," *Jour. Franklin Inst.*, No. 337, 131-145 (2000).
- [10] Casey, S. D., and Sadler, B. M., "New directions in sampling and multi-rate A-D conversion via number theoretic methods," *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2000)*, Vol. 3, 1417-1420 (2000).
- [11] Casey, S. D. "Sampling and reconstruction on unions of non-commensurate sampling lattices via complex interpolation theory," *1999 Intl. Workshop on Sampling Theory and Applications*, pp. 48-51 (1999).

- [12] Casey, S. D., and Sadler, B. M., "Sampling on unions of non-commensurate lattices via complex interpolation theory," *IEEE Int. Conf. Acoustics, Speech, and Sig. Proc. (ICASSP '99)*, Vol. 3, pp. 1641-1644 (1999).
- [13] Casey, S. D., "Modulation and sampling techniques for multichannel deconvolution," *Journal of Inverse and Ill-Posed Problems*, Vol. 6, No. 1, 23-65 (1998).
- [14] Sadler, B. M., and Casey, S. D., "On pulse interval analysis with outliers and missing observations," *IEEE Transactions on Signal Processing*, Vol. 46, No. 11, 2990-3003 (1998).
- [15] Casey, S. D. "Sampling techniques for multichannel deconvolution," *1997 Intl. Workshop on Sampling Theory and Applications*, pp. 279-284 (1997).
- [16] Casey, S. D., Berenstein C. A., and Walnut, D. F., "Exact multichannel deconvolution on radial domains," *IEEE Int. Conf. Acoustics, Speech, and Sig. Proc. (ICASSP '97)*, Vol. 3, pp. 1865-1868 (1997).
- [17] Casey, S. D., "Sampling issues in Fourier analytic vs. number theoretic methods in parameter estimation," invited paper for the *31st Annual Asilomar Conference on Signals, Systems, and Computers*, 6 pp. (1997).
- [18] Lake, D., Sadler, B., and Casey, S., "Detecting regularity in minefields using collinearity and a modified Euclidean algorithm," *Proc. SPIE*, Vol. 3079, pp. 234-241 (1997).
- [19] Casey, S. D., and Sadler, B. M., "Modifications of the Euclidean algorithm for isolating periodicities from a sparse set of noisy measurements," *IEEE Trans. Sig. Proc.*, Vol. 44, No. 9, pp. 2260-2272 (1996).
- [20] Casey, S. D., and Sadler, B. M., "Number theoretic methods in parameter estimation," *IEEE Conf. Stat. Sig. and Array Proc. (SSAP '96)*, pp. 406-409 (1996).
- [21] Sadler, B. M., and Casey, S. D., "Frequency estimation via sparse zero crossings," *IEEE Int. Conf. Acoustics, Speech, and Sig. Proc. (ICASSP '96)*, Vol. 5, pp. 2990-2993 (1996).
- [22] Sadler, B. M., and Casey, S. D., "PRI analysis from sparse data via a modified Euclidean algorithm," *29th Annual Asilomar Conference on Signals, Systems, and Computers*, pp. 1147-1151 (1995).
- [23] Casey, S. D., and Sadler, B. M., "A modification of the Euclidean algorithm for isolating periodicities from a sparse set of noisy measurements," *IEEE Int. Conf. Acoustics, Speech, and Sig. Proc. (ICASSP '95)*, Vol. 3, pp. 1764-1767 (1995).
- [24] Casey, S. D., and Walnut, D. F., "Systems of convolution equations, deconvolution, Shannon sampling, and the wavelet and Gabor transforms," *SIAM Review*, Vol. 36, No. 4, 537-577 (1994).
- [25] Casey, S. D., "Deconvolution and Gabor and wavelet signal analysis," *Proceedings of the EFTF*, pp. 262-274 (1991).
- [26] Pellegrino, J., Sadler, B., and Casey, S., "A residue number system for wide band acousto-optic spectrum analysis," *Proc. IEEE, USS*, pp. 285-290 (1985).

[27] **Additional References**

- [28] Bartlett, M. S., "The spectral analysis of point processes," *Journal of the Royal Statistical Society B*, Vol. 25, No. 2, pp. 264–280 (1963).
- [29] Berenstein, C. A., and Yger, A., "Effective Bezout identities in $\mathbb{Q}[z_1, \dots, z_n]$," *Acta Mathematica*, Vol. 166, pp. 69–120 (1991).
- [30] Berenstein, C. A., Gay, R., and Yger A., "Inversion of the local Pompeiu transform," *J. Analyse Math.*, Vol. 54, pp. 259–70 (1990).
- [31] Berenstein, C. A., and Gay, R., "Le problem de Pompeiu local," *J. Analyse Math.*, Vol. 52, pp. 133–166 (1990).
- [32] Berenstein, C. A., and Taylor, B. A., "Discrete approximations of deconvolution operators," *Transactions of the Seventh Army Conference on Applied Mathematics and Computing*, pp. 479–489 (1990).
- [33] Berenstein C. A., and Patrick, E. V., "Exact deconvolution for multiple convolution sensors – an overview plus performance characterizations for imaging sensors," *Proceedings of the IEEE (Special Issue on Multidimensional Signal Processing)*, Vol. 78, No. 4, pp. 723–734 (1990).
- [34] Berenstein, C. A. and Yger, A., "Analytic Bezout identities," *Advances in Mathematics*, Vol. 10, pp. 51–74 (1989).
- [35] Berenstein, C. A., and Yger, A., "Le probleme de la deconvolution," *J. Funct. Anal.*, Vol. 54, pp. 113–60 (1983).
- [36] Berenstein, C. A., Yger, A., and Taylor, B. A., "Sur quelques formules explicites de deconvolution," *Journal of Optics (Paris)*, Vol. 14, pp. 75–82 (1983).
- [37] Bloomfield, P., "Spectral analysis with randomly missing observations," *Journal of the Royal Statistical Society B*, Vol. 32, No. 3, pp. 369–380 (1970).
- [38] Bouadani, M., "Advances in operator theory via number theoretic methods: deconvolution of chirp operators for LTI systems," *Masters Thesis, American University*, 15 pp. (2002).
- [39] Daubechies, I., *Ten lectures on wavelets* (CBMS–NSF Conference Series in Applied Mathematics, 61), SIAM, Philadelphia, PA (1992).
- [40] Duffin, R. J., and Schaeffer, A. C., "A class of nonharmonic Fourier series," *Trans. Amer. Math. Soc.*, Vol. 72, pp. 341–366 (1952).
- [41] Fivash, M., "Multichannel deconvolution: a computational study," *scholarly paper for the Masters Degree, American University*, 41 pp. (1995).
- [42] Gabor, D., "Theory of communication," *J. Inst. Elec. Eng.*, Vol. 93, No. 3, pp. 429–457 (1946).
- [43] Gaskill, J. D., *Linear Systems, Fourier Transforms, and Optics*, John Wiley and Sons, New York (1978).

- [44] Heil, C.E., and Walnut, D., "Continuous and discrete wavelet transforms," *SIAM Review*, Vol. 31, No. 4, pp. 628–666 (1989).
- [45] Hörmander, L., *The Analysis of Linear Partial Differential Operators I (Distribution Theory and Fourier Analysis)*, Second Edition, Springer-Verlag, New York (1990).
- [46] Hörmander, L., "Generators for some rings of analytic functions," *Bull. American Math. Soc.*, Vol. 73, pp. 943–949 (1967).
- [47] Ireland, K., and Rosen, M., *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York (1982).
- [48] Jain, A. K., *Fundamentals of Digital Image Processing*, Prentice-Hall, New Jersey (1989).
- [49] Jones, R. H., "Spectral analysis with regularly missed observations," *Annals of Mathematical Statistics*, Vol. 33, pp. 455–461 (1962).
- [50] Kay, S. M., *Fundamentals of Statistical Signal Processing*, Prentice-Hall, New Jersey (1993).
- [51] Kay, S. M., and Sudhaker, R., "A zero crossing-based spectrum analyzer," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, Vol. 34, No. 1, pp. 96–104 (1986).
- [52] Kedem, B., *Time Series Analysis by Higher Order Crossings*, IEEE Press, New York (1994).
- [53] Knuth, D. E., *The Art of Computer Programming, Volume 2: Seminumerical Algorithms (Second Edition)*, Addison-Wesley, Reading, Massachusetts (1981).
- [54] Knuth, D. E., *The Art of Computer Programming, Volume 3: Sorting and Searching*, Addison-Wesley, Reading, Massachusetts (1973).
- [55] Leveque, W. J., *Topics in Number Theory, Volumes 1 and 2*, Addison-Wesley, Reading, Massachusetts (1956).
- [56] Levin, B. Ya., *Lectures on Entire Functions*, American Mathematical Society, Providence, RI (1996).
- [57] Mallat, S., "Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$," *Trans. Amer. Math. Soc.*, Vol. 315, No. 1, pp. 69–87 (1989).
- [58] Marr, D., *Vision*, W. H. Freeman, San Francisco (1982).
- [59] McSweeney, L.A., and Short, K.M., "Detection of periodic behavior in unevenly sampled data," *Proceedings of the Joint Statistical Meeting*, 5 pp. (1998).
- [60] Meyer, Y., "Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs," *Séminaire Bourbaki*, Vol. 662 (1985–86).
- [61] Organ, C., "Generalizations of Shannon sampling theory," *Masters Thesis, American University*, 31 pp. (1997).
- [62] Papoulis, A., *Signal Analysis*, McGraw-Hill, New York (1977).

- [63] Papoulis, A., *Probability, Random Variables, and Stochastic Processes (Third Edition)*, McGraw-Hill, New York (1991).
- [64] Parzen, E., "On spectral analysis with missing observations and amplitude modulation," *Sankhya*, Ser. A, Vol. 25, pp. 383-392 (1963).
- [65] Peterson, D. P., and Middleton, D., "Sampling and reconstruction of wave-number-limited functions in N -dimensional Euclidean spaces," *Inform. and Control*, Vol. 5, pp. 279-323 (1962).
- [66] Port, M., and Zeevi, Y., "The generalized Gabor scheme of image representation in biological and machine vision," *IEEE Tr. on Pattern Anal. and Machine Intelligence*, Vol. 10, No. 4, pp. 452-468 (1988).
- [67] Prenter, P. M., *Splines and Variational Methods*, John Wiley and Sons, New York (1975).
- [68] Schoenberg, I. J., *Cardinal Spline Interpolation* (CBMS-NSF Conference Series in Applied Mathematics, 12), SIAM, Philadelphia, PA (1973).
- [69] Schroeder, M. R., *Number Theory in Science and Communication (Second Edition)*, Springer-Verlag, Berlin (1986).
- [70] Shannon, C. E., "Communication in the presence of noise," *Proc. IRE*, Vol. 37, pp. 10-21 (1949).
- [71] Tretter, S. A., "Estimating the frequency of a noisy sinusoid by linear regression," *IEEE Trans. IT*, Vol. 31, pp. 832-835 (1985).
- [72] Wiener, N. *Extrapolation, interpolation, and smoothing of stationary time series*, MIT Technology Press, Cambridge, MA (1949).
- [73] Wiley, R. G., *Electronic Intelligence: The Analysis of Radar Signals (Second Edition)*, Artech House, Norwood, Massachusetts (1993).